

# New Lower Bounds for van der Waerden Numbers Using Distributed Computing

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## Abstract

This project found improved lower bounds for van der Waerden numbers. The number  $W(k, r)$  is the length of the shortest sequence of colors with  $r$  different colors that guarantees an evenly-spaced subsequence (or arithmetic progression) of the same color of length  $k$ . We applied existing techniques: Rabung's using primitive roots of prime numbers and Herwig et al's Cyclic Zipper Method as improved by Rabung and Lotts. We used much more computing power, 2 teraflops, using distributed computing and larger prime numbers, through over 500 million, compared to 10 million by Rabung and Lotts. We used  $r$  up to 10 colors and  $k$  up to length 25, compared to 6 colors and length 12 in previous work. Our lower bounds on  $W(k, r)$  grow roughly geometrically with a ratio converging down towards  $r$  for  $r$  equals 2, 3, and 4. We conjecture that the exact numbers  $W(k, r)$  grow at the rate  $r$  for large  $k$ .

## 1 Introduction

The sequence of colors **BRRBRRB** (where **B** is blue and **R** is red) does not have an evenly spaced subsequence of length 3 that are the same color. However, if you add a **B** to the end, you get **BRRBRRBB**, which has the same color **B** in positions 1,5, and 9 which are evenly spaced 4 apart. If you add an **R** to the end, you get **BRRBRRBR**, which has **R** at positions 3, 6, and 9. In fact, with only two colors, there is no sequence of length 9 of **B**s and **R**s that does not have a subsequence of 3 evenly spaced of the same color. Van der Waerden's Theorem [vdW27] states that for any number of colors  $r$  and length  $k$ , a long enough sequence always has an evenly spaced subsequence of the same color. The smallest length guaranteed to have an evenly spaced subsequence is called the van der Waerden Number and is written  $W(k, r)$ . For example,  $W(3, 2) = 9$ . Only seven van der Waerden numbers are known exactly. They are shown in Table 1 without the  $>$  symbol.

For the others, all we have are upper and lower bounds. Gowers [Gow01] showed that for amount of colors  $r$  and number evenly spaced  $k$ , the longest string of colors you can have without having  $k$  evenly spaced is less than or equal to  $2^{2^{r-2}2^{k+9}}$ . The formula comes up with a number with more than 101,010,616 digits for  $W(3, 2)$ , even though 9 is enough to guarantee 3 evenly spaced. The upper and lower bounds are very far apart which means we need to narrow the gap between them. The few exact van der Waerden numbers we know are much closer to the lower bounds than the upper bounds.

This project came up with better lower bounds. This project did not find upper bounds as there are few methods to find upper bounds for specific van der Waerden Numbers, apart from Gowers' formula and checking every possibility, which is impractical because there are too many. We did not try to find any new exact van der Waerden Numbers, as the last two exact numbers found,  $W(6, 2)$  and  $W(4, 3)$ , required SAT Solvers and computers designed especially for the purpose.

This research topic is part of Ramsey Theory, which studies how big a structure has to be to guarantee that some property holds. For example, Ramsey Numbers specify how big a party has to be to guarantee a group of  $k$  people that all know each other or  $k$  people that do not know each other. Ramsey Theory, including the study of arithmetic progressions, has important applications. Work on this topic led to Green and Tao's [GT08] proof that there are arbitrarily long arithmetic progressions of prime numbers. It also has applications in computer science such as fast matrix multiplication.

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## 2 Methodology

We used Berkeley Open Infrastructure For Network Computing (BOINC), to distribute the work among our volunteers' computers. We were able to get over two teraflops of computing power, or about two billion floating point operations per second, during 12 months. Two computers checked each prime in case one of them made a mistake. We only accepted the result if the computers agreed on the answers. There were a total of 516 volunteers and 1760 computers in 53 countries. We created both Linux and Windows versions. We wrote the program in C++, because it is a fast programming language. Over time, as the primes got bigger, they needed more memory. We had to make the program use less memory by taking all the numbers in the array of powers (see next paragraph) modulo 2520, the least common multiple of the first 10 numbers. That way, we could check all colors from 2 to 10 in parallel.

Here is how the program works. Take a prime number  $p$  (shown in parentheses in Table 2) and a primitive root of that number. For example, let  $p$  equal 11. See that  $W(4, 2)$ -length 4, 2 colors has 11 in parentheses. Let's use the primitive root 2. 2 is a primitive root of 11 because its powers up to  $2^{10}$  [2, 4, 8, 16, 32, 64, 128, 256, 512, 1024] modulo 11 (the remainder when dividing by 11) are all distinct. They equal [2, 4, 8, 5, 10, 9, 7, 3, 6, 1] which we color red, blue, red, blue. Now all we have to do is reorder this is sequence, getting us [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], or BRBBBBRRRBR. We can add the color 11, which should be blue. Rabung [Rab79], using a method attributed to Folkman, proved that under certain conditions we can concatenate 3 more copies of this 11-term sequence while avoiding 4 evenly spaced of the same color. We can add a 34th term, so we will. The existence of this sequence of 34 proves that  $W(4, 2)$  is more than 34. In fact,  $W(4, 2) = 35$  (Table 1).

Rabung showed that you only have to check for arithmetic progressions that are spaced one apart. To see this, suppose there is an arithmetic progression  $a, a + d, a + 2d \dots$  in one of the colors. Multiply through by  $d^{-1}$ , where  $d^{-1}$  is a number such that  $d * d^{-1} = 1 \pmod p$  and get  $ad^{-1}, ad^{-1} + 1, ad^{-1} + 2 \dots$ . We used Rabung's method exhaustively for primes up to 500 million.

We used the Cyclic Zipper Method for up to length 18 and 10 colors. We checked all primes up to 40 million, using code shared by Rabung and Lotts [RL12]. Zipping is a complex method to double the length of a sequence by interleaving it with itself. We found 4 zips, which were  $W(11, 4)$ ,  $W(7, 8)$ ,  $W(8, 8)$ , and  $W(4, 10)$ .

## 3 Results

We have improved many lower bounds and searched much greater  $r$  and  $k$  than previous work, as can be seen in Tables 1-3. These new and improved lower bounds are shown in bold. The lower bounds that are not in bold are the best known of previous work: Rabung and Lotts [RL12], Herwig et al [HHvLvM07], Kouril and Paul [KP08], Landman et al [LRC05], Landman and Robertson [LR14], and Rabung [Rab79]. We checked the number of colors,  $r$ , up to 10, and the length of the subsequence trying to be avoided,  $k$ , up to 23. All of our lower bounds for  $k > 12$  and  $r > 6$  are new, and we improved on 8 existing lower bounds.

Our lower bounds on  $W(k, 2)$  grow roughly geometrically as  $k$  increases with a ratio  $\frac{W'(k, 2)}{W'(k-1, 2)}$  (with  $W'(k, r)$  being this project's lower bounds) that seems to oscillate between 2 and 2.7 when  $k > 14$ , which is shown in Figure 1. The ratio  $\frac{W'(k, 3)}{W'(k-1, 3)}$  seems to hover around 3 when  $r$ , the number of colors, is 3. The ratio  $\frac{W'(k, 4)}{W'(k-1, 4)}$  is around 4 or 5 where  $r$  is 4. The averages of the last three values shown in Figure 1 have ratios for each number of colors for  $r = 2, 3$ , and 4 are 2.3, 3.1, and 4.2, respectively.

These averages are around  $r$  in each case. Therefore, we conjecture that the lower bounds for exact numbers  $W'(k, r)$  also grow at the rate  $r$  for large  $k$ . In fact, we can make a stronger conjecture, given that the lower bounds produced by Rabung's method and Cyclic Zipping exactly match all seven exact numbers except for  $W(3, 2)$  and  $W(3, 3)$ . In fact, Kouril and Paul [KP08] found that every sequence giving a lower bound of 1131 for  $W(6, 2)$  has the same structure as cyclic zipping with the prime 113. Furthermore, Kouril [Kou12] showed that every sequence giving a lower bound of 292 for  $W(4, 3)$  has the same structure as produced by Rabung's method with the prime 97. Therefore, we also conjecture that the exact numbers  $W(k, r)$  also grow at the rate  $r$  for large  $k$  based on this empirical evidence.

There is another argument for this conjecture. Rabung's method colors the  $n$ th entry with color  $\log_\rho n \pmod r$ , where  $\rho$  is a primitive root of  $p$  and  $\log_\rho n$  is the discrete logarithm defined as  $m$  such that  $\rho^m = n \pmod p$ . Therefore, if  $\log_\rho n = \log_\rho n + 1 = \log_\rho n + 2 \pmod r$ , which is a monochromatic arithmetic progression of spacing 1 and length 3, then  $\log_\rho cn = \log_\rho cn + c = \log_\rho cn + 2c \pmod r$ , which is a monochromatic arithmetic progression of spacing  $c$ . As a result, sequences produced by Rabung's method are full of arithmetic progressions of length  $k-1$  of every possible spacing, which is a reason to think they are optimal (possibly with zipping) lower bounds for  $W(k, r)$ . Furthermore, this is a property that sequences not generated using Rabung's method (possibly with zipping) are unlikely to have.

**Conjecture.**  $\lim_{k \rightarrow \infty} \frac{W(k, r)}{W(k-1, r)} = r$ .

There are several lower bounds formulas that have this pattern of growth rates with ratio  $r$ . Berlekamp [Ber68] showed that if  $p$  is prime then  $W(p+1, 2) \geq p(2^p - 1)$ . Szabó [Sza90] found a lower bound for  $W(k, 2) \geq \frac{2^{k-1}}{ek}$ . Landman

and Robertson [LR14] Theorem 2.21 states that for  $p \geq 5, q$  primes,  $W(p+1, q) \geq p(q^p - 1) + 1$ . Graham [Gra07] conjectures that  $W(k, 2) \leq 2^{k^2}$ ; our conjecture is stronger.

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Table 1: Lower Bounds for  $W(k, r)$ 

	2 colors	3 colors	4 colors	5 colors	6 colors	7 colors	8 colors	9 colors	10 colors
Length 3	9	27	76	>170	>223		>83		>383
Length 4	35	293	>1,048	>2,254	>9,778	>5,800	>9,940	>11,128	>28,147
Length 5	178	>2,173	>17,705	>98,740	>63,473	>121,973	>223,173	>405,149	>553,805
Length 6	1,132	>11,191	>91,131	>540,025	>816,981	>1,589,846	>2,707,086	>4,304,706	>11,269,856
Length 7	>3,703	>48,811	>420,217	>1,381,687	>7,465,909	>16,466,695	>35,618,893	>60,413,155	>142,307,167
Length 8	>11,495	>238,400	>2,388,317	>10,743,258	>57,445,718	>119,524,728	>428,424,207	>1,092,421,520	>2,190,170,228
Length 9	>41,265	>932,745	>10,898,729	>79,706,009	>458,062,329	>1,271,890,265	>3,697,406,985	>3,829,814,793	
Length 10	>103,474	>4,173,724	>76,049,218	>542,694,970	>2,615,305,384	>4,308,441,994			
Length 11	>193,941	>18,603,731	>329,263,781	>3,046,508,211	>4,787,268,611				
Length 12	>638,727	>79,134,144	>1,536,435,264	>5,265,995,472					
Length 13	>1,642,309	>251,282,317	>5,683,410,589						
Length 14	>3,118,350	>669,256,082							
Length 15	>8,523,047	>2,250,960,279							
Length 16	>16,370,086	>7,180,474,606							
Length 17	>46,397,777								
Length 18	>91,079,252								
Length 19	>250,546,915								
Length 20	>526,317,462								
Length 21	>1,409,670,741								
Length 22	>2,582,037,634								
Length 23	>6,206,141,987								

> means the actual number is unknown but we know it is more than that number. The lower bounds that are bold are new or have been improved by this project.

Table 2: Primes Used to Find Lower Bounds for  $W(k, r)$ 

	2 colors	3 colors	4 colors	5 colors	6 colors	7 colors	8 colors	9 colors	10 colors
Length 3			37				<b>41</b>		<b>191</b>
Length 4	11	97	349	751	3,259	<b>1,933</b>	<b>3,313</b>	<b>3,709</b>	<b>4,691Z</b>
Length 5	11ZZ		2,213Z		3,967ZZ	<b>30,493</b>	<b>55,793</b>	<b>101,287</b>	<b>138,451</b>
Length 6	113Z		9,133Z			<b>317,969</b>	<b>541,417</b>	<b>860,941</b>	<b>2,253,971</b>
Length 7	617			230,281	622,159Z	<b>2,744,449</b>	<b>2,968,241Z</b>	<b>10,068,859</b>	<b>23,717,861</b>
Length 8	821Z	34,057	85,297ZZ	1,534,751	8,206,531	<b>17,074,961</b>	<b>30,601,729Z</b>	<b>156,060,217</b>	<b>312,881,461</b>
Length 9	2,579Z	116,593	1,362,341	9,963,251	<b>57,257,791</b>	<b>158,986,283</b>	<b>462,175,873</b>	<b>478,726,849</b>	
Length 10	11,497	463,747	8,449,913	<b>60,299,441</b>	<b>290,589,487</b>	<b>478,715,777</b>			
Length 11	9,697Z	1,860,373	<b>16,463,189Z</b>	<b>304,650,821</b>	<b>478,726,861</b>				
Length 12	29,033Z	7,194,013	<b>139,675,933</b>	<b>478,726,861</b>					
Length 13	<b>136,859</b>	<b>20,940,193</b>	<b>473,617,549</b>						
Length 14	<b>239,873</b>	<b>51,481,237</b>							
Length 15	<b>608,789</b>	<b>160,782,877</b>							
Length 16	<b>1,091,339</b>	<b>478,698,307</b>							
Length 17	<b>2,899,861</b>								
Length 18	<b>5,357,603</b>								
Length 19	<b>13,919,273</b>								
Length 20	<b>27,700,919</b>								
Length 21	<b>70,483,537</b>								
Length 22	<b>122,954,173</b>								
Length 23	<b>282,097,363</b>								

The numbers above are the primes we used to find the lower bounds. Results with the cyclic zipper method are shown above. Z=zippped once, ZZ=zippped twice. The lower bounds that are bold are new or have been improved by this project.

Table 3: References for Lower Bounds for  $W(k, r)$ 

	2 colors	3 colors	4 colors	5 colors	6 colors	7 colors	8 colors	9 colors	10 colors
Length 3	[Chv70]	[Chv70]	[BO79]	[HvM09]	[SAT16]		*		*
Length 4	[Chv70]	[Kou12]	[Rab79]	[Rab79]	[Rab79]	*	*	*	*
Length 5	[SS78]	[MHN09]	[HHvLvM07]	[HvM09]	[HHvLvM07]	*	*	*	*
Length 6	[KP08]	[MHN09]	[HHvLvM07]	[HvM09]	[SAT16]	*	*	*	*
Length 7	[Rab79]	[SAT16]	[SAT16]	[RL12]	[RL12]	*	*	*	*
Length 8	[HHvLvM07]	[HHvLvM07]	[RL12]	[RL12]	[RL12]	*	*	*	*
Length 9	[HHvLvM07]	[RL12]	[RL12]	[RL12]	*	*	*	*	
Length 10	[Rab79]	[RL12]	[RL12]	*	*	*			
Length 11	[RL12]	[RL12]	*	*	*				
Length 12	[RL12]	[RL12]	*	*					
Length 13	*	*	*						
Length 14	*	*							
Length 16	*	*							
Length 17	*								
Length 18	*								
Length 19	*								
Length 20	*								
Length 21	*								
Length 22	*								
Length 23	*								

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The asterisks (\*) represent the lower bounds that this project found.

